

Curvature & global optima

1 / Definitions and graphs: concavity and pseudoconcavity

Let C be convex, $x \neq y$ in C , $0 < t < 1$, and $z = (1-t)x + ty$. Gradients require differentiability. Definitions below describe the **maximum side**; for minima apply them to $-f$ (page 3).

Concavity $\Rightarrow$ Pseudoconcavity $\Rightarrow$ Quasiconcavity

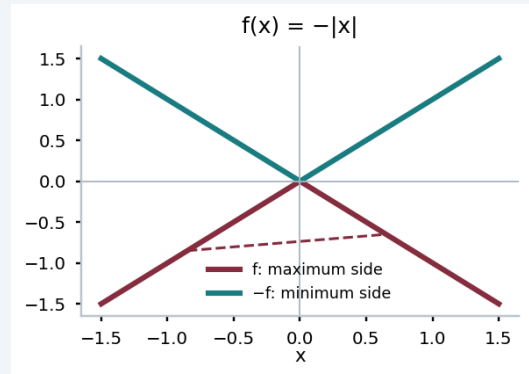
For differentiable f on a convex domain. Reverse implications fail in general.

1. Concavity

Definition: $f(z) \geq (1-t)f(x) + tf(y)$.

Meaning. The graph lies on or above every chord.

Optimum. Every local maximum is global. If $\nabla f(x^*) = 0$, x^* is a global maximum; it need not be unique.



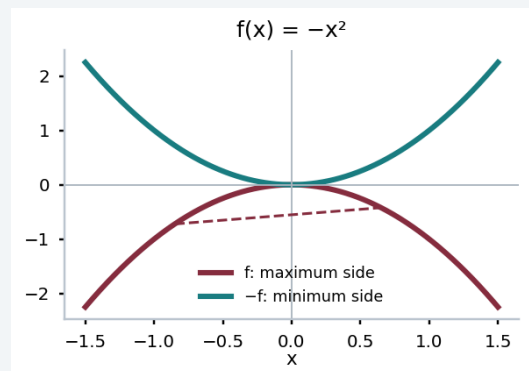
Concave, but not strictly concave: straight sections. Unique peak here; not differentiable at 0.

2. Strict concavity

Definition: $f(z) > (1-t)f(x) + tf(y)$.

Meaning. Every interior point of a chord lies strictly below the graph.

Optimum. A maximum, if attained, is unique. A stationary point is the unique global maximum.



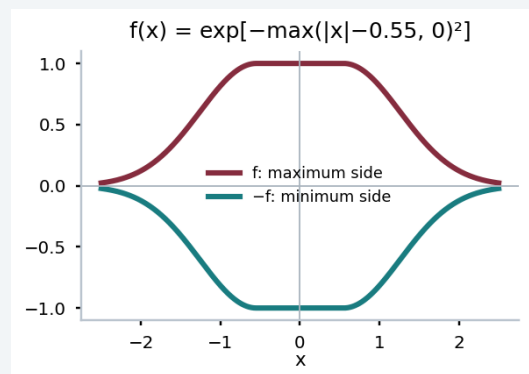
Unique global maximum at $x = 0$; the reflected curve has a unique minimum.

3. Pseudoconcavity

Definition: $f(y) > f(x)$
 $\Rightarrow \nabla f(x)^T(y-x) > 0$.

Meaning. Every better point lies in a direction with a positive first-order slope.

Optimum. Every stationary point is a global maximum. Every local maximum is global. Multiple maxima are possible.



A flat top of global maxima. Pseudoconcave but neither concave nor strictly pseudoconcave.

Strictness & weaker conditions

2 / Definitions and graphs: strict pseudoconcavity and quasiconcavity

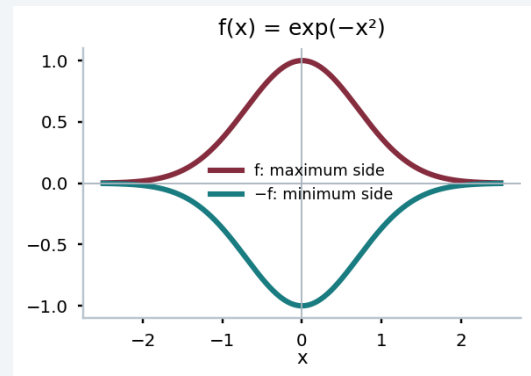
The examples isolate useful differences. A function can belong to several classes at once. **Strict** does not mean that its second derivative must be strictly signed at every point.

4. Strict pseudoconcavity

Definition ($x \neq y$): $f(y) \geq f(x)$
 $\Rightarrow \nabla f(x)^T(y-x) > 0$.

Meaning. Even a distinct point with the same value must lie in a direction with a positive slope.

Optimum. A stationary point is the unique global maximum. Any attained maximum is unique.



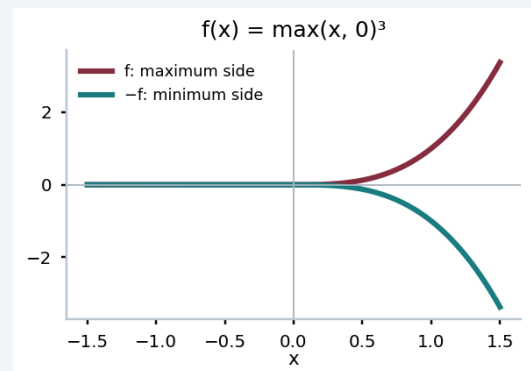
Unique peak at 0. Strictly pseudoconcave but not concave on $\mathbb{R}$: the tails bend upward.

5. Quasiconcavity

Definition: $f(z) \geq \min\{f(x), f(y)\}$.

Meaning. Between two points, the value never falls below the worse endpoint. Upper contour sets are convex.

Optimum. Stationarity alone is insufficient. A non-strict local maximum need not be global. A strict local maximum is global.



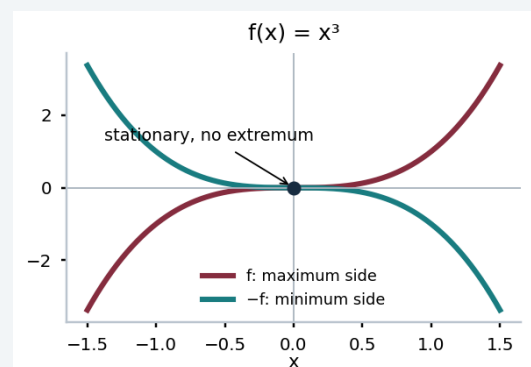
The lower plateau contains stationary local maxima that are not global. Quasiconcave, not pseudoconcave.

6. Strict quasiconcavity

Definition: $f(z) > \min\{f(x), f(y)\}$.

Meaning. Between distinct endpoints, the value is strictly above the worse endpoint.

Optimum. Any local maximum is the unique global maximum. But a stationary point need not be any kind of maximum!



Strictly quasiconcave and strictly quasiconvex. At 0, $f' = 0$, but there is no extremum.

Which condition do I use?

3 / Maximum versus minimum • local versus global • constraints

1. Mirror the definitions for minima

Use the same x , y , z and t as on page 1. The convex versions reverse the relevant inequalities:

Maximum side	Minimum side: direct definition
Concave / strictly concave	Convex / strictly convex: $f(z) \leq / < (1-t)f(x) + tf(y)$
Pseudoconcave	Pseudoconvex: $f(y) < f(x) \Rightarrow \nabla f(x)^T(y-x) < 0$
Strictly pseudoconcave	Strictly pseudoconvex: $f(y) \leq f(x) \Rightarrow \nabla f(x)^T(y-x) < 0; x \neq y$
Quasiconcave / strictly so	Quasiconvex / strictly so: $f(z) \leq / < \max\{f(x), f(y)\}$

2. What is sufficient for a global optimum?

Interior point: $\nabla f(x^*) = 0$ + pseudoconcavity $\Rightarrow$ global maximum; pseudoconvexity $\Rightarrow$ global minimum. The strict pseudo versions give uniqueness. Concavity / convexity are stronger sufficient assumptions.

Quasi alone: even strict quasiconcavity / quasiconvexity does not make stationarity sufficient (x^3 at 0).

Implications, for differentiable functions:

strict concavity $\Rightarrow$ strict pseudoconcavity $\Rightarrow$ strict quasiconcavity;

concavity $\Rightarrow$ pseudoconcavity $\Rightarrow$ quasiconcavity.

Each strict class implies its non-strict version. The reverse implications fail in general.

3. Hessian: distinguish one point from the whole domain

At an interior stationary point of a C^2 function: **H negative definite $\Rightarrow$ strict local maximum**; positive definite $\Rightarrow$ strict local minimum. A singular semidefinite H is inconclusive.

On an open convex domain: **H(x) negative semidefinite everywhere $\Leftrightarrow$ concavity**; positive semidefinite everywhere $\Leftrightarrow$ convexity. Definite everywhere is sufficient for strict curvature, but not necessary: $-x^4$ is strictly concave although $f''(0) = 0$.

4. Constraints: first-order condition over feasible directions

On a convex feasible set C , a differentiable local maximum satisfies

$\nabla f(x^*)^T(y-x^*) \leq 0$ for every $y \in C$ (minimum: ≥ 0).

With pseudoconcavity this is sufficient for a global maximum; with strict pseudoconcavity it is unique. Reverse for minimization. For $\max f$ subject to convex $g_i \leq 0$ and affine equalities, KKT conditions imply this inequality; pseudoconcavity of f makes them sufficient. A constraint qualification is needed to guarantee KKT necessity.

Nonconvex feasible set: objective curvature alone is insufficient. For smooth equality constraints, use the reduced Hessian of the Lagrangian for local tests (as in your definiteness sheet).

5. Existence is a separate question

Continuous f + nonempty compact $C \Rightarrow$ a maximum and minimum exist. Strict curvature gives uniqueness only if the relevant optimum is attained. Example: $-e^x$ is strictly concave on $\mathbb{R}$ but has no maximum. A maximum is strict if every other nearby feasible point has a lower value; global means comparison with every feasible point.